

上覆单相弹性层饱和地基上刚性条形基础竖向振动的混合边值问题

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摘 要: 根据 Biot 动力控制方程, 运用 Fourier 积分变换技术, 并按照混合边值条件和连续条件建立了上覆单相弹性层饱和地基上刚性条形基础竖向振动的对偶积分方程, 并将其退化到完全饱和地基的情形。通过引进正交多项式将对偶积分方程化为线性代数方程组, 从而得到了上覆单相弹性层的饱和地基上刚性条形基础的竖向振动规律。通过算例分析得到, 单相弹性层的厚度对动力柔度系数有着较大的影响, 在单相弹性层厚度较小时 (小于条形基础半宽的 0.1), 动力柔度系数曲线与完全饱和的基本重合; 完全饱和地基上刚性基础的竖向振动是上覆弹性层厚度 $H_n=0$ 的特例。

关 键 词: 动力柔度系数; 条形刚性基础; 混合边界条件;

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Mixed boundary value problem of vertical vibration of a rigid foundation on elastic soils with saturated substratum

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Abstract: Based on the Biot's dynamic equations, the vertical vibration of a rigid strip foundation resting on partially saturated soil subgrade which is composed of a dry elastic layer and a saturated substratum is studied. The analysis relied on the use of Fourier integral transform techniques and a pair of dual integral equations governing the vertical vibration of the rigid foundation is listed under the consideration of mixed boundary value condition and the continuity condition. The results are reduced to the case for saturated half-space. The set of dual integral equations are converted to linear equations by means of infinite series of orthogonal functions—the Jacobi polynomials. Consequently, the dynamic compliance C_v for partially saturated soils versus the dimensionless frequency b_0 is evaluated. Numerical results indicate that: the thickness has great effect on the dynamic coefficient, and as the thickness of the elastic stratum is less than the tenth half width of the strip foundation, the curve of the dynamic compliance versus the dimensionless frequency basically match together with the saturated half-space; and the saturated half space is the special case of the partially saturated soil subgrade (the thickness of the superimposed elastic soil H_n is equal to 0).

Key words: dynamic compliance coefficient; strip rigid foundation; mixed boundary-value condition

1 前 言

饱和土是由土骨架和充满其中孔隙的水相构成的二相介质, Biot^[1]建立了流体饱和多孔介质波动方程 (1962 年 Biot 补充了该理论^[2]), 使得真正地研究饱和地基上基础振动特性成为了可能。从那以后, 众多学者致力于研究饱和多孔介质的动力问题。Halpern 等^[3,4]应用 Fourier 积分变换技术,

给出了饱和多孔介质弹性半空间在表面作用稳态简谐牵引力下的解, 并探讨了矩形刚性基础的动力响应。同一时期, Kassir 等^[5]也研究了多孔饱和半空间上条形刚性基础竖向振动的混合边值问题。Rajapakse 等^[6]运用动力 Green 函数方法探讨了层状饱和地基上条形刚性基础的动力响应。在实际工程中, 地下水位以下的土可视为饱和土, 而地下水位以上的则视为弹性干土。Philippacopoulos^[7]扩展了

Hallpern 等工作,成功的解析求解了上覆单相弹性层的多孔饱和半空间上刚性圆板的竖向振动。陈胜利^[8]得到了上覆单相弹性层的饱和地基上轴对称圆板的竖向振动规律。李庆金等^[9]采用 Fourier 积分变换计算得到了饱和地基上刚性条形基础的竖向振动规律,并退化到弹性半空间进行了对比。

本文从 Biot 方程出发,采用 Fourier 积分变换技术,按照层间连续条件和混合边值条件,建立了一组描述刚性条形基础竖向振动的对偶积分方程,并利用 Jacobi 正交多项式进一步将其转化为一组线性方程组,通过求解方程组得到了上覆单相弹性层的饱和地基上刚性条形基础的竖向振动规律。

2 基本动力方程

不考虑土和水的重量,同时认为土是各向同性的,水是不可压缩的。本文研究的是平面应变问题,认为基础是无限长条的,基础宽度为 $2b$ 。以水平向为 x 轴,竖向向为 z 轴建立坐标,并把坐标原点置于地基表面,即地基表面 $z=0$,如图 1 所示。

本文基本上采用了力学中常用的常规符号及习惯表示法。应力的正负号按土力学的习惯,即压为正,拉为负。

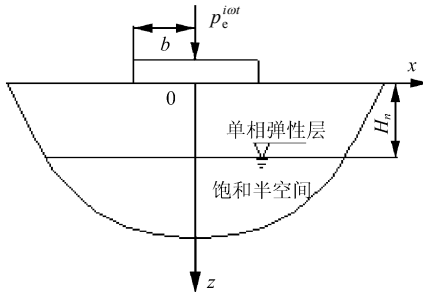


图 1 力学模型与坐标系

Fig.1 Description of the model and coordinate system

2.1 平面应变下弹性层的动力方程

弹性层的波动方程为

$$\left. \begin{aligned} \frac{\partial \sigma_{xL}}{\partial x} + \frac{\partial \tau_{xzL}}{\partial z} + \rho_L \frac{\partial^2 u}{\partial t^2} &= 0; \\ \frac{\partial \tau_{xzL}}{\partial x} + \frac{\partial \sigma_{zL}}{\partial z} + \rho_L \frac{\partial^2 w}{\partial t^2} &= 0 \end{aligned} \right\} \quad (1)$$

式中 u, w 分别为土骨架的水平 and 竖向位移; σ_{xL}, σ_{zL} 分别为水平向和竖向的正应力; ρ_L 为土骨架密度。

应力和位移的关系为

$$\left. \begin{aligned} \sigma_{xL} &= -\frac{2G_L(1-\mu_L)}{1-2\mu_L} \frac{\partial u}{\partial x} - \frac{2G_L\mu_L}{1-2\mu_L} \frac{\partial w}{\partial z}; \\ \sigma_{zL} &= -\frac{2G_L\mu_L}{1-2\mu_L} \frac{\partial u}{\partial x} - \frac{2G_L(1-\mu_L)}{1-2\mu_L} \frac{\partial w}{\partial z}; \\ \tau_{xzL} &= -G_L \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \end{aligned} \right\} \quad (2)$$

式中 G_L 为土的剪切模量; μ_L 为泊松比。

2.2 平面应变下饱和土半空间的动力方程

饱和土半空间的基本动力方程为

$$\left. \begin{aligned} \frac{\partial \sigma'_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + \frac{\partial p_f}{\partial z} + \rho \frac{\partial^2 u}{\partial t^2} + \rho_f \frac{\partial^2 w_x}{\partial t^2} &= 0; \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma'_z}{\partial z} + \frac{\partial p_f}{\partial z} + \rho \frac{\partial^2 w}{\partial t^2} + \rho_f \frac{\partial^2 w_z}{\partial t^2} &= 0; \\ -\frac{\partial p_f}{\partial x} &= \frac{\rho_f g}{k_d} \frac{\partial w_x}{\partial t} + \rho_f \frac{\partial^2 u}{\partial t^2} + \frac{\rho_f}{n} \frac{\partial^2 w_x}{\partial t^2}; \\ -\frac{\partial p_f}{\partial z} &= \frac{\rho_f g}{k_d} \frac{\partial w_z}{\partial t} + \rho_f \frac{\partial^2 w}{\partial t^2} + \frac{\rho_f}{n} \frac{\partial^2 w_z}{\partial t^2}; \\ -\frac{\partial^2 w_x}{\partial x \partial t} - \frac{\partial^2 w_z}{\partial z \partial t} &= \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 w}{\partial z \partial t} \end{aligned} \right\} \quad (3)$$

式中 w_x, w_z 分别为水相对于土骨架的水平 and 竖向位移; σ'_x, σ'_z 分别为水平向和竖向的有效正应力; p_f 为超静孔隙水压力; ρ_s, ρ_f 分别为土颗粒和水的密度; n 为土体的孔隙率; ρ 为饱和土的质量密度, $\rho = (1-n)\rho_s + n\rho_f$ 。

应力和位移方程为

$$\left. \begin{aligned} \sigma'_x &= -\frac{2G(1-\mu)}{1-2\mu} \frac{\partial u}{\partial x} - \frac{2G\mu}{1-2\mu} \frac{\partial w}{\partial z}; \\ \sigma'_z &= -\frac{2G\mu}{1-2\mu} \frac{\partial u}{\partial x} - \frac{2G(1-\mu)}{1-2\mu} \frac{\partial w}{\partial z}; \\ \tau_{xz} &= -G \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \end{aligned} \right\} \quad (4)$$

其中, G 为土的剪切模量; μ 为泊松比。

$$\text{又有 } \sigma_x = \sigma'_x + p_f; \sigma_z = \sigma'_z + p_f \quad (5)$$

3 动力方程的求解

在简谐荷载作用下,位移、应力和超静孔隙水压力可以表示为

$$\left. \begin{aligned} u &= \bar{u} e^{i\omega t}; w = \bar{w} e^{i\omega t}; w_x = \bar{w}_x e^{i\omega t}; \\ w_z &= \bar{w}_z e^{i\omega t}; p_f = \bar{p}_f e^{i\omega t}; \sigma_{xL} = \bar{\sigma}_{xL} e^{i\omega t}; \\ \tau_{xzL} &= \bar{\tau}_{xzL} e^{i\omega t}; \sigma_{zL} = \bar{\sigma}_{zL} e^{i\omega t}; \\ \sigma'_x &= \bar{\sigma}'_x e^{i\omega t}; \tau_{xz} = \bar{\tau}_{xz} e^{i\omega t}; \sigma'_z = \bar{\sigma}'_z e^{i\omega t}; \\ \sigma_x &= \bar{\sigma}_x e^{i\omega t}; \sigma_z = \bar{\sigma}_z e^{i\omega t} \end{aligned} \right\} \quad (6)$$

式中 ω 为圆频率。

Fourier 变换记为

$$\phi^* = \Re \phi = \int_{-\infty}^{\infty} \phi e^{i\zeta x} dx$$

联立式(1)、式(2)，并经傅立叶变换可得弹性层的应力和位移的解：

$$\left. \begin{aligned} i\bar{u}^*(\zeta, z) &= -A_1 \zeta H_L e^{q_L z} - A_2 e^{F_L z} / \zeta - \\ &\quad B_1 \zeta H_L e^{-q_L z} - B_2 e^{-F_L z} / \zeta ; \\ \bar{w}^*(\zeta, z) &= A_1 q_L H_L e^{q_L z} + A_2 e^{F_L z} / F_L - \\ &\quad B_1 q_L H_L e^{-q_L z} - B_2 e^{-F_L z} / F_L ; \\ \bar{e}^*(\zeta, z) &= A_1 e^{q_L z} + B_1 e^{-q_L z} ; \\ i\bar{\tau}_{xzL}^*(\zeta, z) &= 2A_1 \zeta q_L H_L G_L e^{q_L z} - \\ &\quad 2B_1 \zeta q_L H_L G_L e^{-q_L z} + A_2 (F_L / \zeta + \zeta / F_L) G_L e^{F_L z} - \\ &\quad B_2 (F_L / \zeta + \zeta / F_L) G_L e^{-F_L z} ; \\ \sigma_{zL}^*(\zeta, z) &= -2G_L [A_1 (q_L^2 H_L + \mu_L / (1 - 2\mu_L)) e^{q_L z} + \\ &\quad A_2 e^{F_L z} + B_1 (q_L^2 H_L + \mu_L / (1 - 2\mu_L)) e^{-q_L z} + B_2 e^{-F_L z}] \end{aligned} \right\} \quad (7)$$

$$\text{式中 } \bar{e} = \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} ; F_L^2 = \zeta^2 - \frac{\omega^2}{G_L} \rho_L ;$$

$$D_L = -\frac{\omega^2 (1 - 2\mu_L) \rho_L}{2G_L (1 - \mu_L)} ; q_L^2 = \zeta^2 + D_L ;$$

$$H_L = \frac{1}{q_L^2 - \zeta^2} = \frac{1}{D_L} .$$

同理，联立式(3)~(5)，并经 Fourier 变换可得饱和半空间的应力和位移的解：

$$\left. \begin{aligned} i\bar{u}^*(\zeta, z) &= -A_0 \zeta H e^{-qz} - B_0 \zeta M e^{-\zeta z} - C_0 e^{-Fz} / \zeta ; \\ \bar{w}^*(\zeta, z) &= -A_0 q H e^{-qz} - B_0 \zeta M e^{\zeta z} - C_0 e^{-Fz} / F ; \\ \bar{p}_f^*(\zeta, z) &= A_0 E e^{-qz} + B_0 e^{-\zeta z} ; \\ \bar{e}^*(\zeta, z) &= A_0 e^{-qz} ; \\ i\bar{\tau}_{xz}^*(\zeta, z) &= -2A_0 \zeta q H G e^{-qz} - 2B_0 \zeta^2 M G e^{-\zeta z} - \\ &\quad C_0 (F / \zeta + \zeta / F) G e^{-Fz} ; \\ \bar{\sigma}_z^*(\zeta, z) &= -2G [A_0 (q^2 H + \mu / (1 - 2\mu)) e^{-qz} + \\ &\quad B_0 \zeta^2 M e^{-\zeta z} + C_0 e^{-Fz}] + A_0 E e^{-qz} + B_0 e^{-\zeta z} \end{aligned} \right\} \quad (8)$$

$$\text{式中 } \bar{\sigma}_z^*(\zeta, z) \text{ 为 } \sigma_z \text{ 的 Fourier 变换。} c = \frac{nk_d \omega}{gni - k_d \omega} ;$$

$$\bar{e} = \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} ; F^2 = \zeta^2 - \frac{\omega^2}{G} (c \rho_f + \rho) ;$$

$$E = \frac{\omega^2 \rho_f (c + 1)}{Dc} ; D = \frac{\omega^2 (1 - 2\mu) (2c \rho_f - c \rho)}{2cG(1 - \mu)} ; q^2 = \zeta^2 + D ;$$

$$H = \frac{1}{q^2 - \zeta^2} = \frac{1}{D} ; M = \frac{c + 1}{G(\zeta^2 - F^2)} = \frac{c + 1}{\omega^2 (c \rho_f + \rho)} .$$

4 混合边值条件

采用放松的边界条件，即接触面上无切向应力（不存在摩擦力），基底和半空间表面底支点间可自由滑动，这时接触面上只给定了竖向位移，而没有规定切向位移，并且饱水半空间表面是透水的，侧位移、应力连续条件及边界条件为

$$\left. \begin{aligned} \bar{\tau}_{xzL}(x, 0) &= 0 , \quad -\infty < x < \infty ; \\ [\bar{w}(x, H_n)]_L &= \bar{w}(x, H_n) , \quad -\infty < x < \infty ; \\ [\bar{u}(x, H_n)]_L &= \bar{u}(x, H_n) , \quad -\infty < x < \infty ; \\ \bar{\tau}_{xzL}(\zeta, H_n) &= \bar{\tau}_{xz}(\zeta, H_n) , \quad -\infty < x < \infty ; \\ \bar{\sigma}_{zL}(\zeta, H_n) &= \bar{\sigma}_z(\zeta, H_n) , \quad -\infty < x < \infty ; \\ \bar{p}_f(\zeta, 0) &= 0 , \quad -\infty < x < \infty ; \\ \bar{\sigma}_{zL}(x, 0) &= 0 , \quad |x| > b ; \\ [\bar{w}(x, 0)]_L &= \bar{w}_0 , \quad |x| \leq b \end{aligned} \right\} \quad (9)$$

5 对偶积分方程的建立与求解

结合式(7)和式(8)，对式(9)分别进行 Fourier 变换，可得到表面竖向位移与应力的关系：

$$\bar{w}^*(\zeta, 0) = f(\zeta) \bar{\sigma}_z^*(\zeta, 0) \quad (10)$$

根据式(9)，并进行 Fourier 反变换得到：

$$\begin{aligned} \bar{w}(x, 0) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{w}^*(\zeta, 0) e^{-i\zeta x} d\zeta = \\ &= \frac{1}{\pi} \int_0^{\infty} f(\zeta) \bar{\sigma}_z^*(\zeta, 0) \cos(\zeta x) d\zeta \end{aligned} \quad (11)$$

又因

$$\begin{aligned} \bar{\sigma}_z(x, 0) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\sigma}_z^*(\zeta, 0) e^{-i\zeta x} d\zeta = \\ &= \frac{1}{\pi} \int_0^{\infty} \bar{\sigma}_z^*(\zeta, 0) \cos(\zeta x) d\zeta \end{aligned} \quad (12)$$

$$\text{令 } \tilde{\sigma}_z^*(\tilde{\zeta}, 0) = \bar{\sigma}_z^*(\zeta, 0) / G , \tilde{f}(\tilde{\zeta}) = G f(\zeta) ,$$

$$\tilde{w}_0 = \frac{\bar{w}_0}{b} , \tilde{\zeta} = \frac{\zeta}{b} , x = b\tilde{x} , \text{ 则式 (11) 和式 (12) }$$

可化成以下对偶积分方程：

$$\left. \begin{aligned} \int_0^{\infty} \tilde{f}(\tilde{\zeta}) \tilde{\sigma}_z^*(\tilde{\zeta}, 0) \cos(\tilde{\zeta} \tilde{x}) d\tilde{\zeta} &= \pi \tilde{w}_0 b^2 , \quad 0 \leq |\tilde{x}| \leq 1 \\ \int_0^{\infty} \tilde{\sigma}_z^*(\tilde{\zeta}, 0) \cos(\tilde{\zeta} \tilde{x}) d\tilde{\zeta} &= 0 , \quad |\tilde{x}| > 1 \end{aligned} \right\} \quad (13)$$

设 $\tilde{\sigma}_z^*(\tilde{x}, 0)$ 能表达为 Jacobi 多项式组成的级数：

$$\tilde{\sigma}_z^*(\tilde{x}, 0) = \frac{\tilde{w}_0 b}{\sqrt{1 - \tilde{x}^2}} \sum_{n=0}^{\infty} A_n G_n \left(0, \frac{1}{2}, \tilde{x}^2 \right) , \quad 0 \leq |\tilde{x}| \leq 1 \quad (14)$$

利用如下的关系式：

$$\int_0^1 \frac{1}{\sqrt{1 - \tilde{x}^2}} G_n \left(0, \frac{1}{2}, \tilde{x}^2 \right) \cos \tilde{\zeta} \tilde{x} d\tilde{x} = \frac{\pi}{2} J_{2n}(\tilde{\zeta})$$

$$\text{式中 } G_n(\alpha, \gamma, x) = {}_2F_1(-n, \alpha + n; \gamma; x) .$$

对式 (14) 进行 Fourier 变换:

$$\tilde{\sigma}_z^*(\tilde{\zeta}, 0) = \pi \tilde{w}_0 b^2 \sum_{n=0}^{\infty} A_n J_{2n}(\tilde{\zeta}), \quad 0 \leq \tilde{x} \leq 1 \quad (15)$$

则式 (14) 每一项均自动满足。把式 (15) 代入式 (13), 得到:

$$\sum_{n=0}^{\infty} A_n \int_0^{\infty} \tilde{f}(\tilde{\zeta}) J_{2n}(\tilde{\zeta}) \cos(\tilde{\zeta} \tilde{x}) d\tilde{\zeta} = 1 \quad (16)$$

$$\text{又有 } \cos \tilde{\zeta} \tilde{x} = \sum_{k=0}^{\infty} J_{2k}(\tilde{\zeta}) G_k(0, \frac{1}{2}, \tilde{x}^2)$$

代入式 (16) 得到:

$$\sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} A_n C_{2n, 2k} \right) G_k \left(0, \frac{1}{2}, \tilde{x}^2 \right) = 1 \quad (17)$$

$$\text{式中 } C_{2n, 2k} = \int_0^{\infty} \tilde{f}(\tilde{\zeta}) J_{2n}(\tilde{\zeta}) J_{2k}(\tilde{\zeta}) d\tilde{\zeta}.$$

令式 (17) 中同阶的 Jacobi 多项式的系数为零,

注意到 $k=0$ 时, $G_0 \left(0, \frac{1}{2}, \tilde{x}^2 \right) = 1$ 。并取式 (15) 中级数

的前 $(n+1)$ 项计算, 式 (17) 可转化为一组线性代数方程组, 通过求解方程组可得到 $A_0, A_1, \dots, A_n$, 则可由下述方法求出基础底面反力的合力 R 。

$$R = \int_{-b}^b \tilde{\sigma}_z(x, 0) dx = \pi G \tilde{w}_0 b^2 A_0 = \pi G \tilde{w}_0 b A_0.$$

$$\text{现定义}^{[10]}: \quad R = \pi G \frac{1}{C_v} \tilde{w}_0 \quad (18)$$

则饱和地基的动力柔度系数 C_v 可记为

$$C_v = \frac{1}{b A_0}$$

记 $f_1 = \text{Re } C_v$, $f_2 = \text{Im } C_v$, 则可由动力柔度系

数求得基础的刚度为 $K = \frac{f_1}{f_1^2 + f_2^2}$, 阻尼为 $C =$

$$-\frac{f_2}{(f_1^2 + f_2^2)b_0}, \quad b_0 \text{ 为无量纲频率, } b_0 = b\omega \sqrt{\frac{\rho}{G}}.$$

在上述推导过程中, 令 $H_n=0$, 则问题转变为研究饱和半空间上刚性条形基础的竖向振动。

6 算例

算例中地基的物理力学参数为: 对于单相弹性层, $G_L=35 \text{ MPa}$, $\mu_L=0.45$, $\rho_L=17.23 \text{ kN/m}^3$; 饱和半空间, $n=0.35$, $G=35 \text{ MPa}$, $k_d=10^{-5} \text{ m/s}$, $\rho_f=10.0 \text{ kN/m}^3$, $\rho_s=26.5 \text{ kN/m}^3$, $\mu=0.25$ 。条形刚性基础的半宽为 2 m , 通过计算得到了不同弹性层的情况下饱和地基上刚性条形基础竖向振动的动力柔度系数 C_v 随 b_0 的变化曲线 (图 2)。

从图中可以看出, 在给定的参数下, 弹性层的厚度对动力柔度系数有着较显著的影响, 在无量纲频率较小时, 动力柔度系数受无量纲频率的影响较大, 在较大的情况下, 曲线趋于平缓, 受无量纲频率的影响很小。当 $H_n=0 \text{ m}$ 时, 就退化为完全饱和

半空间的情况, 与文献[9]的结果一致。且有, 当 $H_n=0.2 \text{ m}$ ($H_n/b=0.1$) 时, 动力柔度系数曲线与完全饱和半空间 ($H_n=0$) 的值基本重合。

7 结 语

本文从 Biot 动力方程出发, 应用 Fourier 积分变换, 通过解对偶积分方程得到了上覆单相弹性层

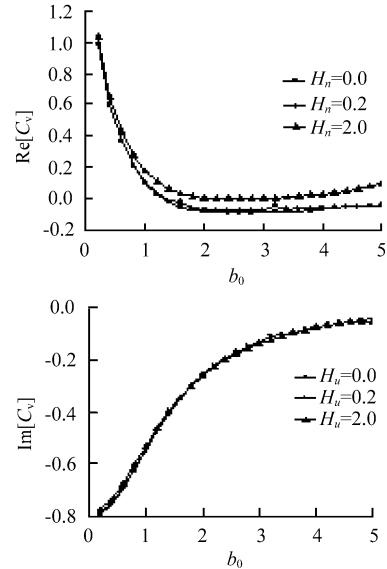


图 2 不同的单相弹性层厚度时的动力柔度系数 C_v
Fig.2 The dynamic compliance coefficients C_v versus dimensionless frequency b_0

的饱和地基上条形刚性基础的竖向振动, 考察了不同的弹性层厚度对动力柔度系数的影响。分析表明, 完全饱和地基上刚性条形基础的竖向振动即为本文中单相弹性层厚度为 0 时的特殊情形; 在给定的算例参数下, 当弹性层厚度小于条形基础半宽的 0.1 时, 可忽略上覆弹性层对竖向振动影响, 作为完全饱和地基进行处理。

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